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Suppression of energy fluctuations in the classical counterpart of quantum models

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We study the relation between the energy-level statistics of a quantal system and the integrability of its classical counterpart, derived using coherent states. Energy levels of any given distribution are reproduced using models based either on the harmonic oscillator or on an su(2) algebraic model. The energy-level fluctuations seen in the quantum models are greatly suppressed in their classical counterparts. This is opposite to what is usually seen, namely, the quantum system suppresses sensitivity on the initial conditions of its classical chaotic counterpart.

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The subject of "quantum chaos" has been studied intensively in recent years, although there is still no commonly accepted definition of it. The main difficulty lies in the fact that the most evident characteristic of classical chaos, which is extreme sensitivity of the motion of a system to its initial conditions, has no direct counterpart in the quantum analog. Lacking such a direct connection between classical chaos and the corresponding quantum system, attention has been paid to finding quantum observables which correlate with the appearance of classical chaos [1–6] or in the study of quantum nonintegrability [7, 8] in which the focus is on the question of quantum-classical correspondence. In the former direction, mainly with many numerical experiments, quantum level statistics were studied extensively and a strong correlation was found between classical chaotic behavior and quantum Gaussian-orthogonal-ensemble (GOE) level statistics. Despite some counterexamples [9–11], many practitioners in this field continue to refer to systems with GOE statistics as chaotic quantum systems. The view is that quantum-level statistics of a certain kind, possibly but not necessarily GOE, are a signature of classical chaos reflected in quantum systems [12].

In this article, we ask the opposite question: What is the reflection in classical mechanics when the corresponding quantum system obeys some given level statistics? Since we are going to concentrate on quantum level fluctuations, it is convenient to unfold any given spectrum so that on average the density of states is unity. The first task is to find a quantum system which will generate the desired spectrum. Reference [11] used a one-dimensional model with a local potential field. The resulting potential is such that on average it is a harmonic-oscillator potential, but with many fine oscillations. While this provided a counterexample to the use of GOE to define quantum chaos, it suffers, like other examples, from being a purely numerical result. When a model for 900 levels is constructed, one can always ask, "What if I have 9000 levels?" To avoid this reproach, we will here provide an analytical result.

The second task is to find the classical analog of the quantum model. In this paper, we adopt the widely accepted coherent state approach. Two of the simplest textbook examples of coherent states are those using (a) the harmonic oscillator (field coherent state) and (b) the su(2) algebraic model (atomic coherent state). By means of these coherent states, the classical counterparts are well defined. To our surprise, quantum level fluctuations of any kind are greatly suppressed in our models. The resulting classical model is always well behaved. Since there are no levels defined in classical mechanics, what we mean here is that the energy becomes a smooth function in phase space, much smoother than the quantal energy. We wish to emphasize the analogy between this phenomenon and the suppression of sensitivity to initial conditions when one goes from a classical system to the corresponding quantum system.

We begin our discussion by defining our quantum model. Instead of trying to find a model with certain spectral statistics, we "fit" our model to a given spectrum with the level statistics one wants. For a set of $N \gg 1$ levels with any desired type of level-spacing statistics, $E_n,\ n = 0, ..., N-1$, we find the local average density $\rho_s(E_n)$, and then construct the unfolded spectrum with local average density unity:

$$e_{n+1} - e_n = (E_{n+1} - E_n)\rho_s(E_n), \quad e_{N-1} - e_0 \equiv N - 1 \tag{1}$$

where $\rho_s(E_n)$ is the smooth local average level density. The value of e_0 can be fixed by requiring the global average of $e_n - n$ to be zero. Next, we define an interpolating function $f(u)$ such that

$$f(n) = e_n. \tag{2}$$

Obviously, such an interpolating function f always exists, but it is not unique. One way to construct a well-behaved function $f(u)$ is through a discrete Fourier transform applied to the data series $e_n - n$, which is oscillating about zero:

$$\mathcal{F}_k = \sum_{n=0}^{N-1} (e_n - n) e^{2\pi i k n/N}, \quad k = 0, ..., N-1. \tag{3}$$

Then the function $f(u)$ is defined through the inverse Fourier transformation

$$f(u) = u + \frac{1}{N}\Bigg\{ \mathcal{F}_0 + \delta_{N=\text{even}} \mathcal{F}_{N/2} \cos \pi u + 2 \sum_{k=1}^{(N-1)/2} \text{Re}[\mathcal{F}_k e^{-2\pi i u k/N}] \Bigg\}. \tag{4}$$

This function satisfies Eq. (2) exactly, and it is infinitely differentiable everywhere, because it is a finite sum. Since the result only involves linear terms and trigonometric functions, it can be always expanded as convergent Taylor series. (Admittedly, this property will fail in the limit $N \to \infty$ where the Fourier transform can approximate a discontinuous function.) Figure 1 plots $f(u)$ for a 500-level spectrum with GOE statistics. It should be understood that, although we have given a particular way to construct $f(u)$, the details of this function are not important. What will affect the later development are only the values at the integer values of u. The simplest form of $f(u)$ which reproduces the spectrum would be the Lagrange interpolating polynomial; however, the resulting function displays much larger fluctuations than does the spectrum itself. Originally, the function $f(n)$ is only defined at integer points up to N, but this can be extended to all integers. A natural way is to extend the fluctuation $f(n) - n$ to be a periodic function of period N, as implicitly defined by the Fourier transform:

$$f(n) \equiv f(k) - k + n \quad \text{where}\, k = n \,\text{mod}(N). \tag{5}$$

We now construct our first quantum model based on the harmonic oscillator. Instead of taking the Hamiltonian to be $H = \hat{N} + 1/2$, we set

$$H = f(\hat{N}), \tag{6}$$

where $\hat{N}$ is the number operator. This Hamiltonian has the same set of eigenstates $|n\rangle$, $n = 0, ...,$ as the usual harmonic oscillator, but its energy-level spectrum is exactly the given unfolded spectrum reproduced by the function $f(u)$. Without going through the details, we know that the classical counterpart of the quantum model is one dimensional and thus integrable. Thus we have an example whereby one can build a quantum system with any desired level statistics whose classical counterpart is integrable. The GOE statistics, of course, is only a special case.

We now derive the classical model explicitly. For the harmonic oscillator, the coherent states are

$$|\alpha\rangle = \exp(-\tfrac{1}{2}\alpha^*\alpha) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \tag{7}$$

The classical Hamiltonian is the Q representation of H:

$$\mathcal{H} = \langle \alpha | H | \alpha \rangle = \exp(-\alpha^*\alpha) \sum_{n=0}^{\infty} f(n) \frac{(\alpha^*\alpha)^n}{n!}. \tag{8}$$

In making this correspondence, the quantity $\alpha^*\alpha$ is just half the sum of the squares of the coordinate and its conjugate momentum. We will call this u:

$$u = \alpha^*\alpha = \frac{1}{2}(p^2 + q^2). \tag{9}$$

Since $\alpha^*\alpha$ is the classical analog of the operator $a^\dagger a$, u is the classical analogue of $\hat{N}$. In other words, u here has a similar meaning as the argument of the function f used to define the quantum Hamiltonian. In terms of u,

$$\mathcal{H} = \frac{1}{2}(p^2 + q^2) + \sum_{n=0}^{\infty} p_n(u)[f(n) - n], \tag{10}$$

where $p_n(u) = e^{-u}u^n/n!$ is just the Poisson distribution for a variable n with average value u, $\sum_{n=0}^{\infty} n p_n(u) = u$. $f(n) - n$ is the deviation of the given spectrum from a straight line, that is, the fluctuation of the spectrum. On average, it is zero. Note that this result depends *only* on $f(u)$ evaluated at integer values of u and is therefore independent of the particular interpolation procedure leading to Eqs. (3) and (4) above. Compared to H, $\mathcal{H}$ now consists of two parts, $u = (p^2 + q^2)/2$, the usual harmonic-oscillator Hamiltonian, plus a small fluctuation term. The effect of summation over n is to make a local average of $f(n) - n$ around $n = u$ with weight function $p_n(u)$. Because $p_n(u)$ is smooth in n and $[f(n) - n]$ oscillates rapidly around zero, much of the fluctuation is

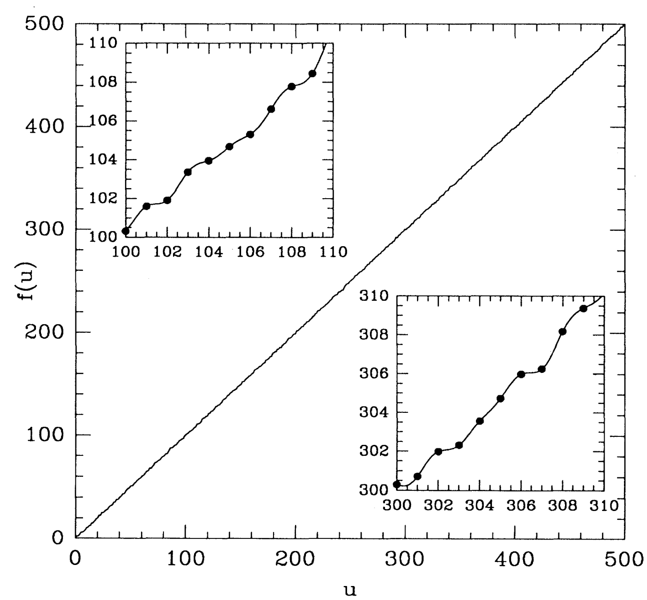

FIG. 1. Function $f(u)$ derived from a set of 500 unfolded energy levels whose level spacings obey GOE statistics. At the integer values of u, $f(u)$ yields exactly the same energy values. Inset: upper and lower parts are magnified regions $u \in (100, 110)$ and $u \in (300, 310)$. Solid dots represent the given spectrum.

washed out, and the resulting fluctuation in the classical Hamiltonian is much smaller than that in the quantum model. In Fig. 2, which compares the function $f(u) - u$ to the $\sum_{n=0}^{\infty} p_n(u)[f(n) - n]$, one can see clearly the filter effect resulting from $p_n(u)$. As a quantitative measure, we calculated the root-mean-square fluctuation shown in Fig. 2, which for the quantum system is 0.46, while for the classical it is 0.05, nearly a factor of 10 smaller. In short, we have found the energy to be a function only of the quantity u, which takes integer values in quantum mechanics and varies continuously in the classical case. The classical energy is a much smoother function of u than is the corresponding quantum-mechanical energy, as illustrated in Fig. 2. Because of this dramatic suppression of the fluctuations, the classical model is much more "harmonic" than the corresponding quantum model.

It should be pointed out that this procedure to deduce the classical Hamiltonian is not reversible, i.e., quantizing $\mathcal{H}$ in the usual way ($[q,p] = i\hbar$) does not always return to H [13]. However, this is not a disturbing fact; it is our philosophy that quantum mechanics should be treated as the fundamental theory from which classical mechanics should be derived, and not vice versa. This point of view is shared by an increasing number of people, as expressed recently by Kleppner [14]. It is also worth mentioning that the quantum Hamiltonian H is nonlocal in that it implies a momentum dependence of the potential function. By using only a local potential in $H = p^2/2 + V(x)$ one would not be able to have any degenerate levels, and thus a spectrum with Poisson-type level-spacing statistics could not be realized. The nonlocality of the current approach makes it possible to realize spectra of arbitrary statistics. The fact that in the present approach, an integrable one-dimensional system can be forced to possess a chaotic spectrum suggests that the spectrum alone may not be sufficient to determine whether a quantum system is chaotic or integrable.

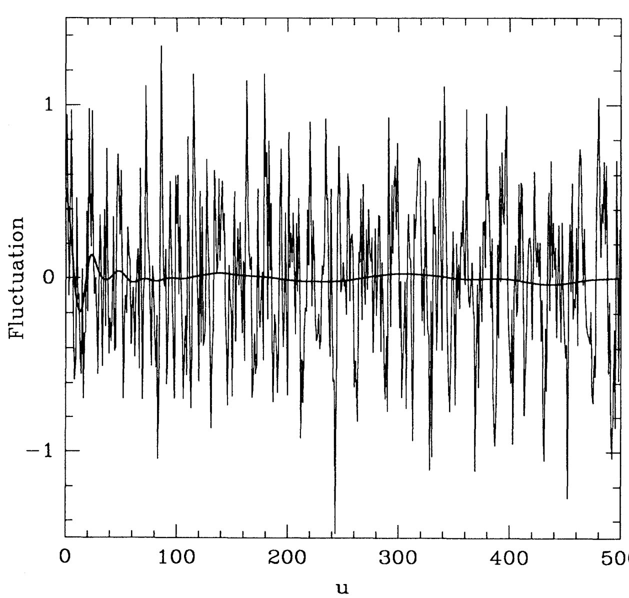

FIG. 2. Fluctuations of quantum and classical Hamiltonians. Light line: $f(u) - u$ represents the quantum fluctuations; heavy line: $\sum_{n=0}^{\infty} p_n(u)[f(n) - n]$, the fluctuation remaining in the classical Hamiltonian.

Similarly, the model of a fixed spin squared can also serve our purpose. We follow the same procedure to define the function $f(u)$. Consider a fixed spin value

$$J = \frac{N-1}{2}. \tag{11}$$

The quantum Hamiltonian is now defined as

$$H = f(J_z + J), \quad J_z = -J, -J+1, \ldots, +J\,, \tag{12}$$

which reproduces the desired spectrum. The classical counterpart, which will be reached when $J \to \infty$, is also one dimensional, and thus integrable.

Following the usual procedure, the su(2) coherent state is defined as [13]

$$|\xi\rangle = \exp(\xi J_+ - \xi^* J_-)|-J\rangle = \left(\frac{\tau}{1+\tau^*\tau}\right)^J \sum_{M=-J}^{J} \sqrt{C_{2J}^{J+M}}\,\tau^M |M\rangle, \tag{13}$$

where

$$\xi = \frac{\theta}{2} e^{-i\phi} \quad , \tag{14}$$

$$\tau = \tan\theta e^{-i\phi} \quad , \tag{15}$$

with θ, ϕ being spherical coordinates on a sphere in S^2 and C_{2J}^{J+M} are the binomial coefficients, the number of ways to select $J+M$ objects from $2J$. A semiquantal Hamiltonian is then defined as [8]

$$\mathcal{H} = \langle\xi|H|\xi\rangle = \frac{(\tau^*\tau)^J}{(1+\tau^*\tau)^{2J}} \sum_{M=-J}^{J} C_{2J}^{J+M} f(M+J)(\tau^*\tau)^M \tag{16}$$

with a conjugate coordinate and momentum pair q, p defined by

$$\frac{1}{\sqrt{2}}(q + ip) = \sqrt{2J} e^{-i\phi} \sin\frac{\theta}{2} \quad . \tag{17}$$

Setting $u = (p^2 + q^2)/2$, one finds

$$\tau^*\tau = \frac{u}{2J - u} \tag{18}$$

and

$$\mathcal{H} = \frac{1}{2}(p^2+q^2) + \sum_{n=0}^{2J} B_n\left(2J, \frac{u}{2J}\right)[f(n) - n], \tag{19}$$

where

$$B_n\left(2J, \frac{u}{2J}\right) = C_{2J}^n \left(1 - \frac{u}{2J}\right)^{2J-n} \left(\frac{u}{2J}\right)^n \tag{20}$$

is the binomial distribution for a variable n with mean value u. $\mathcal{H}$ consists of a major term $\mathcal{H}_0 = (p^2+q^2)/2$, plus a small term which reflects fluctuations. Since $0 \leq (p^2+q^2) \leq 4J$ [see Eq. (17)], $\mathcal{H}_0$ represents a confined harmonic oscillator. The fluctuation term, for the same reasons stated in the preceding section, is much smaller than the quantum fluctuation term $[f(u) - u]$. We note that the problem of extending $f(u)$ outside the basic interval does not arise here. In this case, the Poisson distribution has been replaced by the binomial distribution. It is known that the classical limit for the su(2) model is reached when $J \to \infty$. In that limit, it can be proven that for any finite u, $B_n[2J, u/(2J)] \to p_n(u)$. Thus

$$\lim_{J\to\infty} \mathcal{H}|_{\mathrm{su}(2)} = \mathcal{H}|_{\mathrm{HO}}. \tag{21}$$

In summary, we have used simple one-dimensional models to demonstrate two points. First, that it is possible to construct a model having any specified quantum spectral statistics for which the classical counterpart is integrable. We would like to remark that, although we have chosen the harmonic oscillator and the su(2) algebraic model to demonstrate our idea, other models with higher dimension could also be used. Think of a model with some dynamical symmetry; its spectrum can be labeled by group-chain representations. It is possible to define the Hamiltonian as a function of group-chain subgroup Casimir operators, so that the resulting spectrum obeys a given statistics. If one accepts that models with dynamical symmetry are integrable, then this shows that they can have any type of level statistics. For a discussion of how a change in dynamical symmetry affects level statistics, see [15, 16].

The second point we have made is that spectral fluctuations, which are thought to be closely related to quantum chaos, can be greatly suppressed in the classical counterpart. It is interesting to compare this with the generic behavior that, when one goes from a classical chaotic system to its quantum counterpart, the sensitivity to the initial conditions is suppressed.

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